Homework Ten

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Math 295
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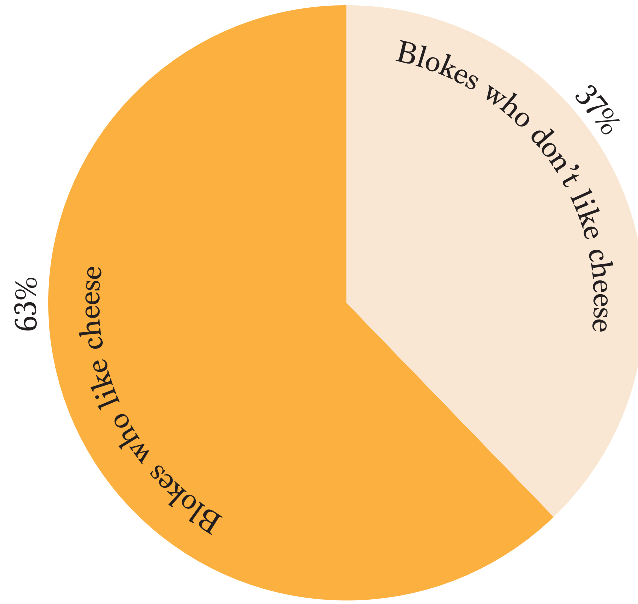
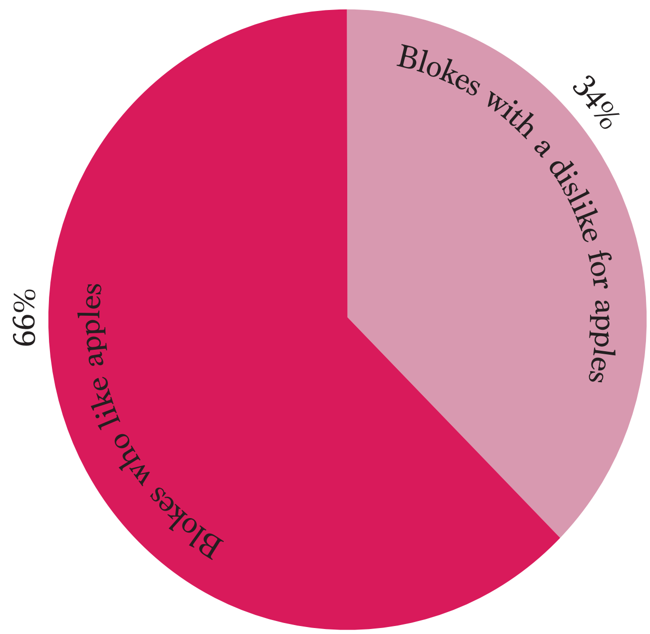
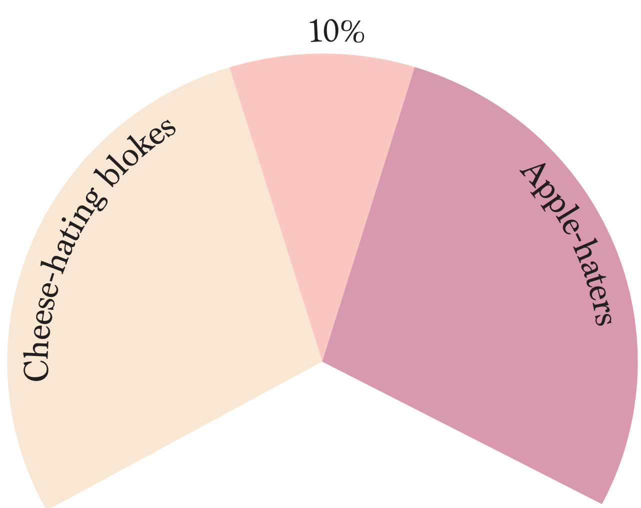
1. It is given that 63% of the British folks like cheese. This would imply that the remaining 37% don't. It's also provided that 66% like apples. This implies a complement of 34% that don't like apples.



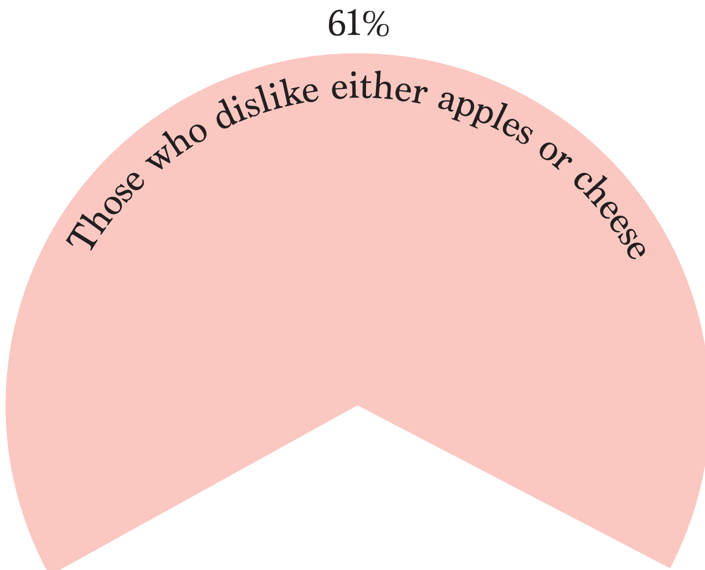


It is said that this overlap of those who dislike both cheese and apples is 10%.

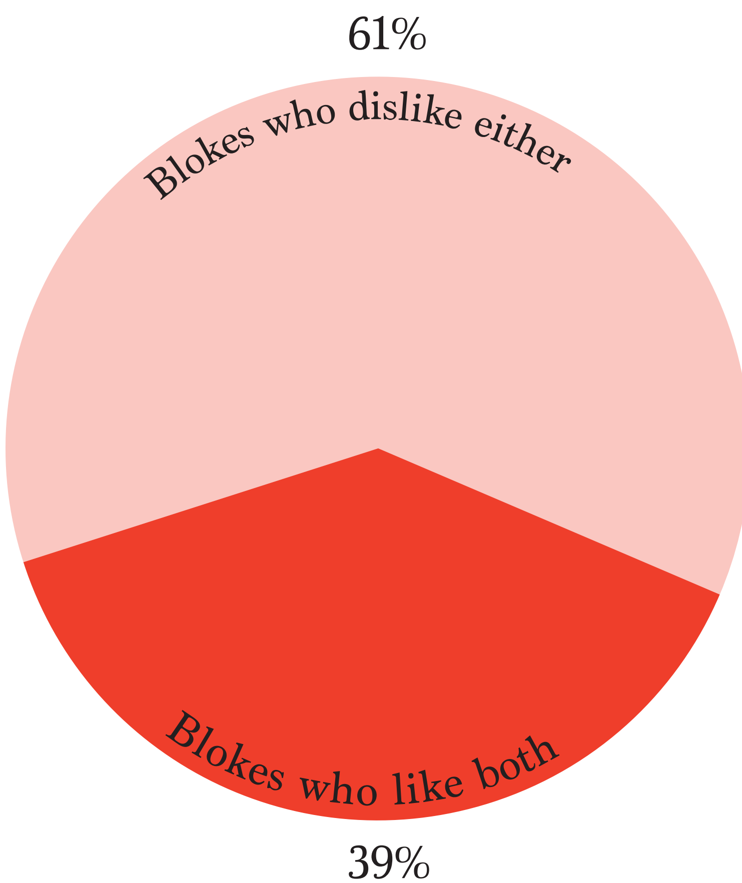


To be solved is how many like both cheese and apples. I will try to minimize the negations and booleans in the semantics.
Derived from the fact that the amount of those that dislike cheese is 37%, those that dislike apples are 34%, and the intercection of those that dislike cheese and apples are 10%, what is the union of those sets?

This must be the sum of 37% and 34%, then minus the redundant overlap of 10%. Thus, the union of those who dislike either is equal to 61%.



Since this is the set of those who dislike anything at all, the remaing part must be those who dislike nothting; in other words, like both apples and cheese. What value of 61% is complementry to 100%? The anwser is 39%.



Thus, 39% of the jolly-olde-British folk like both apples and cheese.

2.

Assuming they are using the Gregorian calendar, there are 366 distinct possible birthdays. In a set of 1800 people these birthdays will have to repeat. To interpret the question as how many people have an ambiguous birthday in common with someone else, the smallest largest set possible is when all birthdays are distrubuted evenly. The remainder is then rounded up to join a larger group. Thus, the smallest set that is the largest is 5.

$$\left\lceil 1{,}800 \div 366 \right\rceil = 5$$

The leap day does not make a significant difference on the quotient. Even excluding the leap day, the answer is still 5.

$$\left\lceil 1{,}800 \div 365 \right\rceil = 5$$

3a. First is to determine how many integers between 1 and 4,000 *are* divisable by three. Every third integer is a multiple of three, thus the amount of factors of three between 1 and 4,000 is just 4,000 ÷ 3, then rounded down.

$$\lfloor 4{,}000 \div 3 \rfloor = 1{,}333$$

The amount of integers up to 4000 that are *not* divisable by three would just be the complement of that.

$$4{,}000 - 1{,}333 = 2{,}667$$

The same goes for multiples of 4. The amount of numbers divaiable by 4 up to 4,000 is an even 1,000.

$$4000 \div 4 = 1000$$

The amount of such numbers not divisable by 4 is simply then 3,000.

$$4000 - 1000 = 3000$$

Now how many numbers *are* divisible by 3 *and* 4? 3 itself is a prime number, and 4 is relativly prime with 3. Thus the least common multiple of 3 and 4 is 12.

$$\mathrm{LCM}(3,4) = 12$$

Thus, all factors of 12 and only factors of 12 will both divisable by 3 and 4. How many of such numbers are there up to 4000?

$$\lfloor 4{,}000 \div 12 \rfloor = 333$$

This is how many numbers *are* divisable by 3 *and* 4 up to 4,000. The negation of this becomes:
How many numbers up to 4,000 are *not* divisable by 3 *or* 4?

The union of these two sets is the sum of the sizes minus the intercestion:
1,333 + 1000 - 333 = 2,000.

3b. We already know that there are exactly 1,000 multiples of 4 up to 4,000. Now how many are divisable by 5?

This is another exact operation:

$$4{,}000 \div 5 = 800$$

Now to determine the intercection of how may are divisble by 4 and 5. Five itself is a prime number, so the least common multiple of 4 and 5 is 20. All factors of 20, and only factors of 20, are divisable by both 4 and 5. So how many factors of 20 exist up to 4,000? Another even division:

$$4{,}000 \div 20 = 200$$

How many factors of 6 there are up to 4,000? This spooky number:

$$\lfloor 4000 \div 6 \rfloor = 666$$

5 and 6 are also relativley prime, so their least common multiple has to be their product: 30. The intercestion of the numbers that are divisable by both 4 and 5 must therefore be:

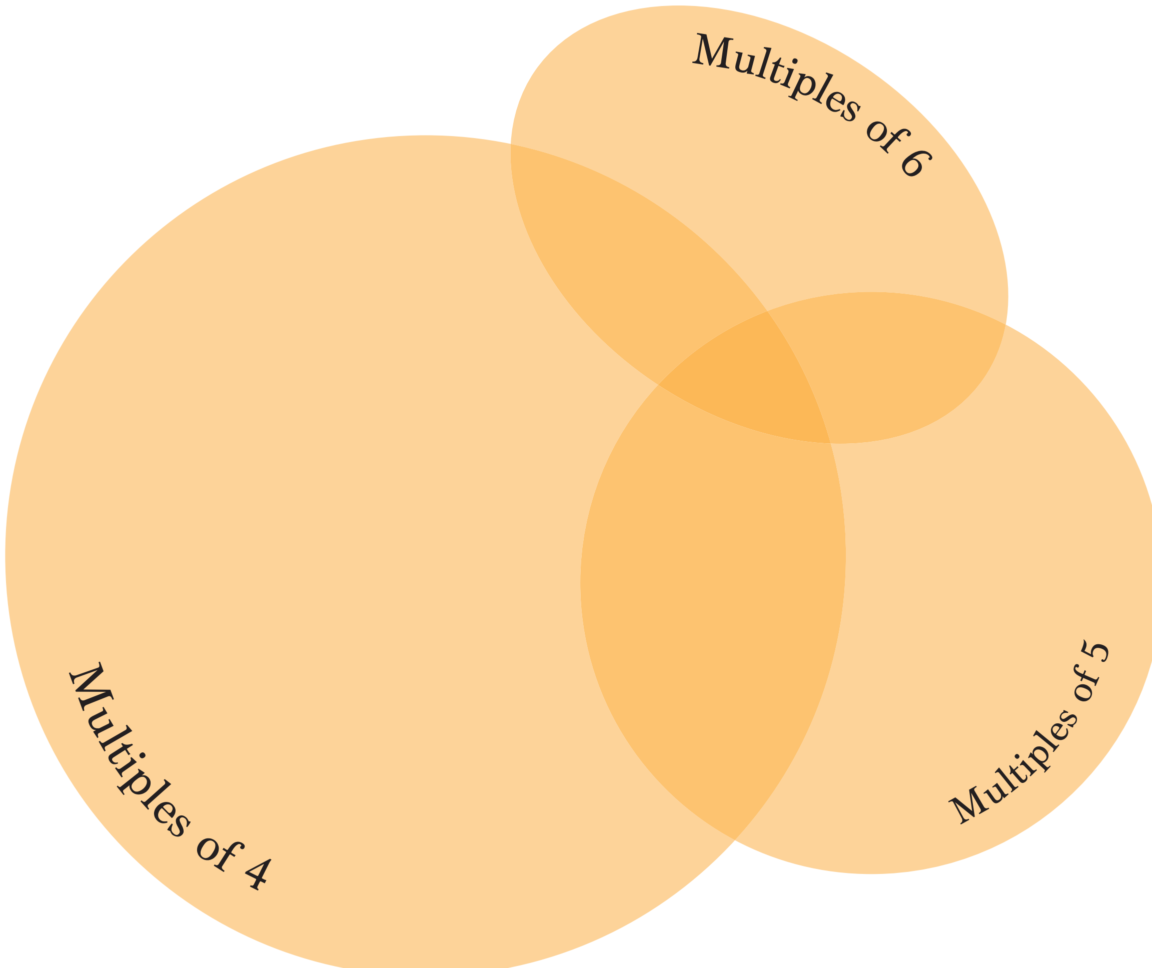
$$\lfloor 4{,}000 \div 30 \rfloor = 133$$

We must also acount for the intercestion of the sets of numbers that are divisable by 4 and by 6. The least common multipe of those numbers in reference is 12. We know from the last probem that there are 333 factors of 12 up to 4,000.

Now we must account for what numbers are simaltaniusly multiples for 4, 5 and 6. The least common multple of 4, 5, and 6 is 60. Again applying our routine division and rounding down:

$$\lfloor 4{,}000 \div 60 \rfloor = 66$$

Fig. 1 One overlap zone removed from each area would emilate the entire triple interection.



Because intercestions are redundant, the cardinality of the union of all three sets must be their sum minus the intercestions. However, eliminating the intesesctions three times would completly eliminate the triple intercection completly. (Figure 1) We must then add back the triple inersection.

$$1{,}000 + 800 + 666 - 200 - 133 - 333 + 66 = 1{,}866.$$

So there, the answer is 1,866.

4. This is provided by the pigeonhole principle which that a set of *n* elements divided into *m* subsets must have multiple elements that share a certian subset. (With *n* being greater than *m*.)

Becuase this set has 2n elements, at least one of the numbers in a n+1 subset will unavoidably be consecetive with another number in the series.

$$\{1 \quad\quad\quad\} \quad \{3 \quad\quad\quad\} \quad \{5 \quad\quad\quad\}$$

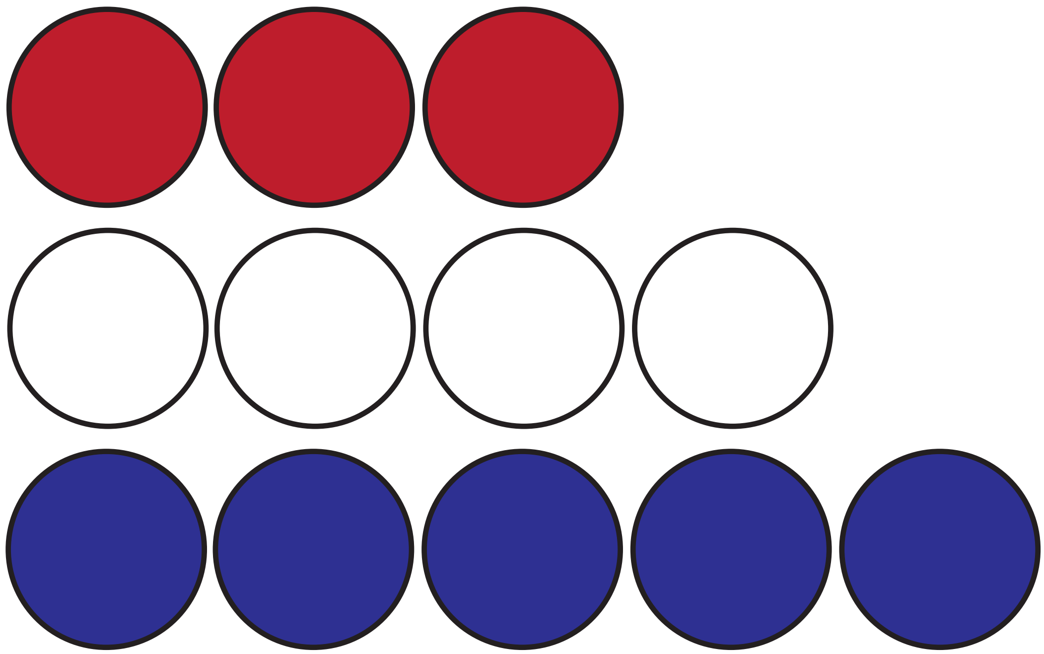
These consecetive integers will surely only have 1 be the sole common divisor.

This relies on the premise that any two concestive integers can not have any factors in common, in other words,"relativly prime." Becuase no prime factor divides into 1, incrementing the number by 1 makes it impossible fot n+1 to have any factors in common with *n*.

This principle is also utilized by the ituitive proof by induction that there are infinitly many prime nubmers. The factorial of any number P is a product of all natural numbers up to P, and thus is a factor of all of them. Simply incremeting this number by one rules out P+1 being divisible by any factor up to P. Becuase P! + 1 can not have any divisors up to P, there must be a larger prime number than P.

$$P! + 1$$

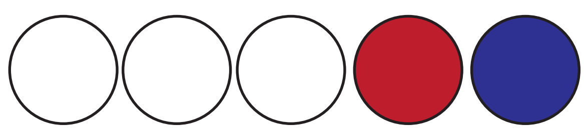
5. Out of the 12 balls is a chosen set of five. How many subsets out of 12 have five elements? 12 choose 5, or 792



The balls are not homegenous however, and must all be counted independently. How many ways can these three white balls be chosen out of four? Just four.

$$\binom{4}{3} = \frac{4!}{1! \times 3!} = 4$$

But of those with five elements, how many have three white balls? Given that there are three out of the five are white, this leaves two more to be chosen. One of the cases is a red ball and a blue ball.



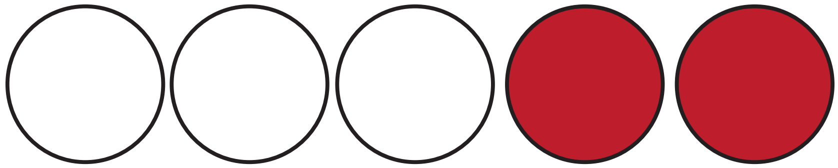
How many incidents are there for this outcome? 3-choose-1 red ball, then times 5-choose-1 blue balls.

$$\binom{3}{1} = \frac{3!}{1! \times 2!} = 3 \qquad \binom{5}{1} = \frac{5!}{1! \times 4!} = 5$$

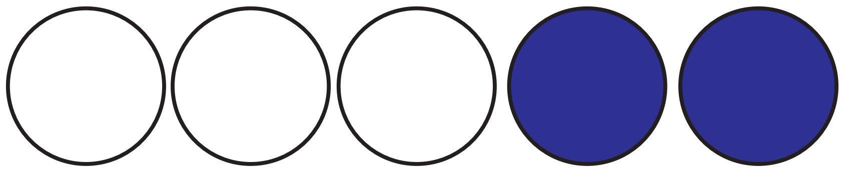
$3 \times 5 = 15$

so 15 possible cases in total for this configuration.

Now for the incedents of two red balls. This would be 3-choose-2, which is 3.



And finally the cases of two blue balls, which would be 5-choose-2. Which equates to 10.



Out of this configuration of white balls, there total combinations are 10 + 3 + 15 = 28

This must be overall multiplied by 4 for each of the other subsets of white balls.

$4 \times 28 = 112$

So thre, there are 112 diferent sets of five balls with three white balls.

6. The first digit has five possible values. The next following digit can be one of the four remaining values. The next one can only be one of the three, then two, and one.

a. The number of possible permutations of the five digits are 5! or 120.

b. Out of these 120 values, the amount of those that are even end in 4 or 2 are:
$\frac{5!}{5} \times 2 = 48$

c.

i. The quality of being divisible by three depends of the sum of the digits being divisible by three. Due to addition being communicative, the sum of 1, 2, 3, 4, 5 will always be
$$\sum_{i=1}^{5} i = 15$$
no matter what order they are. 15 is divisible by three, so every single one of the 120 numbers in the set will be divisible by three (assuming these numbers are base-ten). If these numbers in this set were base-six however, only the numbers that end in the digit 3 would be multiples of three, which there would be $\frac{5!}{5} = 24$ of.

ii. The rule for divisibility by four is even the last two digits of the number is a multiple of four. The only two-digit multiples of four with digits 1 though 5 without any repeated digits are 12, 24, and 32. Within each incidence allows permutations of three remaining elements: 3! or 6. Then times three for each of the instances.
$3! \times 3 = 18$

iii. Out of these numerals, the only trailing digit that can create a value divisible by five is 5. The amount of such numbers are $\frac{5!}{5} = 4! = 24$.